International Journal of Theoretical Physics, Vol. 28, No. 11, 1989

Lie-Symmetry Vector Fields for Linear and Nonlinear Wave Equations

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Received February 16, 1989; revised May 1, 1989

We derive the Lie symmetry vector fields for the linear wave equation $\Box u = 0$ and nonlinear wave equation $\Box u = u^3$. The conformal vector fields for the underlying metric tensor field g are also given. We construct the conservation laws and derive similarity solutions. Furthermore, we perform a Painlevé test for the nonlinear wave equation and discuss whether Lie–Bäcklund vector fields exist.

Lie symmetry vector fields are important in the study of linear and nonlinear evolution equations. With the help of the Lie symmetry vector fields we can construct similarity solutions and conservation laws (Bluman and Cole, 1974; Anderson and Ibragimov, 1979; Ovsiannikov, 1982; Olver, 1986; Steeb and Strampp, 1982; Grauel and Steeb, 1985). Moreover, for relativistic field equations such as the Dirac equation, they play an important role in connection with gauge theory.

In the present paper we give the Lie-symmetry vector field for the linear wave equation

$$\Box u = 0 \tag{1}$$

and the nonlinear wave equation

$$\Box u = u^3 \tag{2}$$

where $\Box = \sum_{i=1}^{3} \partial^2/\partial x_i^2 - \partial^2/\partial x_4^2$. The underlying metric tensor field is given by

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_4 \otimes dx_4 \tag{3}$$

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Equation (1) can be derived from

$$d(*du)=0 \tag{4}$$

where d is the exterior derivative and $*$ the Hodge-star operator (Steeb, 1980).

For Riemannian or pseudo-Riemannian manifolds the Hodge-star operator (duality operation) is defined on differential forms: $*$ is an f-linear mapping and transforms a p-form into its dual $(m-p)$-form ($\dim M = m$). The $*$ operator applied to a p-form defined on an arbitrary Riemannian (or pseudo-Riemannian) manifold with metric tensor field g is given by

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_p}):$$

$$= \sum_{j_1 \cdots j_m = 1}^{m} g^{i_1 j_1} \ldots g^{i_p j_p} \frac{1}{(m-p)!} \frac{g}{\sqrt{|g|}} \varepsilon_{j_1 \cdots j_m} \, dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_m} \tag{5}$$

where $\varepsilon_{j_1 \cdots j_m}$ is the total antisymmetric tensor ($\varepsilon_{1,2,\ldots,m}=+1$), $g \equiv \det(g_{ij})$, and $\sum_j g^{ij} g_{jk} = \delta^i_k$ (Kronecker symbol).

In the present case we have $M = \mathscr{R}^4$ with g given by equation (3) (Minkowski space) and the one-form

$$du = \sum_{j=1}^{4} \frac{\partial u}{\partial x_j} dx_j \tag{6}$$

so that

$$*du = \sum_{j=1}^{4} \frac{\partial u}{\partial x_j} (*dx_j) \tag{7}$$

and

$$\begin{aligned} *dx_1 &= -dx_2 \wedge dx_3 \wedge dx_4 \\ *dx_2 &= -dx_3 \wedge dx_1 \wedge dx_4 \\ *dx_3 &= -dx_4 \wedge dx_1 \wedge dx_2 \\ *dx_4 &= -dx_1 \wedge dx_2 \wedge dx_3 \end{aligned} \tag{8}$$

We then obtain

$$d(*du) = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2}\right) \Omega \tag{9}$$

where

$$\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \tag{10}$$

is the volume element in Minkowski space.

Before studying the Lie-symmetry vector fields it is helpful to find the conformal invariant vector fields $\mathscr{V}(x)$. The conformal invariant vector fields are defined by

$$L_{\mathscr{V}} g = \rho_{\mathscr{V}} g \tag{11}$$

for some $\mathscr{C}^{\infty}$ function $\rho_{\mathscr{V}}$, where $L_{\mathscr{V}}(\cdot)$ denotes the Lie derivative. The conformal invariant vector fields for the metric tensor field (3) are given by

$$\mathscr{T} = \frac{\partial}{\partial x_4} \qquad \rho_{\mathscr{T}} = 0$$

$$\mathscr{P}_i = \frac{\partial}{\partial x_i} \qquad \rho_{\mathscr{P}_i} = 0, \quad i = 1, 2, 3$$

$$\mathscr{R}_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \qquad \rho_{\mathscr{R}_{ij}} = 0, \quad i \neq j, \quad i, j = 1, 2, 3$$

$$\mathscr{L}_i = x_i \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_i} \qquad \rho_{\mathscr{L}_i} = 0, \quad i = 1, 2, 3$$

$$\mathscr{S} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \qquad \rho_{\mathscr{S}} = 2$$

$$\mathscr{I}_4 = 2x_4 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) + (x_1^2 + x_2^2 + x_3^2 + x_4^2) \frac{\partial}{\partial x_4} \qquad \rho_{\mathscr{I}_4} = 4x_4$$

$$\mathscr{I}_i = -2x_i \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} \right) + (x_1^2 + x_2^2 + x_3^2 - x_4^2) \frac{\partial}{\partial x_i} \qquad \rho_{\mathscr{I}_i} = -4x_i, \quad i = 1, 2, 3 \tag{12}$$

The physical interpretation of the given vector fields (Lie group generators) is the following: $\mathscr{T}$ generates time translation; $\mathscr{P}_i$ space translation; $\mathscr{R}_{ij}$ space rotation; $\mathscr{L}_i$ space-time rotations; and $\mathscr{S}$ the uniform dilatations ($x \to \varepsilon x$, $\varepsilon > 0$). $\mathscr{I}_4$ is the conjugation of $\mathscr{T}$ by the inversion in the unit hyperboloid Q: $(x) \to (x)/(x_1^2 + x_2^2 + x_3^2 - x_4^2)$, and the $\mathscr{I}_i$ are the conjugations of the $\mathscr{P}_i$ by Q.

Let us now consider the Lie symmetry vector fields for the linear wave equation (1). We adopt the jet bundle technique (Steel and Strampp, 1982). Within this technique we consider the submanifold

$$F \equiv u_{11} + u_{22} + u_{33} - u_{44} = 0 \tag{13}$$

where $u_{x_1} \equiv u_1$, $u_{x_1 x_1} \equiv u_{11}$, $u_{x_2} \equiv u_2$, and so on. Together with $F=0$ we consider all its differential consequences. This means

$$F_1 \equiv u_{111} + u_{122} + u_{133} - u_{144}, \qquad F_2 \equiv u_{112} + u_{222} + u_{233} - u_{244}$$

and so on. Let

$$V_v = \left(-\sum_{i=1}^{4} a_i(x, u) u_i + g(x, u) \right) \frac{\partial}{\partial u} \tag{14}$$

be the vertical Lie symmetry vector field of V. The invariance requirement is expressed as

$$L_{\bar{V}_v} F \triangleq 0 \tag{15}$$

where $L_{\bar{V}_v}(\cdot)$ denotes the Lie derivative and $\triangleq$ stands for the restriction to solutions of equation (1). $\bar{V}$ is the extended (or prolongated) vector field of V. For our case the Lie-symmetry vector fields $V(x, u)$, for equation (1) with metric g, are given by

$$V(x, u) = \mathscr{V}(x) + V'(x, u) \tag{16}$$

where $V'(x, u)$ is the component of the generator that considers the transformation of the field. We find $L_{\bar{V}_v} F = \rho_{\bar{V}_v} F$, where

$$\begin{aligned}
&T = \frac{\partial}{\partial x_4} && \rho_{\bar{T}} = 0 \\
&P_i = \frac{\partial}{\partial x_i} && \rho_{\bar{P}_i} = 0, \quad i = 1, 2, 3 \\
&R_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} && \rho_{\bar{R}_{ij}} = 0, \quad i \neq j, \quad i = 1, 2, 3 \\
&L_i = x_i \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_i} && \rho_{\bar{L}_i} = 0, \quad i = 1, 2, 3 \\
&S = x_4 \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - u \frac{\partial}{\partial u} && \rho_{\bar{S}} = -3 \\
&I_4 = 2x_4 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - u \frac{\partial}{\partial u} \right) && \\
&\qquad + (x_1^2 + x_2^2 + x_3^2 + x_4^2) \frac{\partial}{\partial x_4} && \rho_{\bar{I}_4} = -6x_4 \\
&I_i = 2x_i \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - u \frac{\partial}{\partial u} \right) && \\
&\qquad + (x_1^2 + x_2^2 + x_3^2 - x_4^2) \frac{\partial}{\partial x_i} && \rho_{\bar{I}_i} = -6x_i, \quad i = 1, 2, 3
\end{aligned} \tag{17}$$

The commutation relations are presented in Table I. The commutation relations are given between the generators in the first column, with the the generators in the first row of Table I. They do form a Lie algebra, as they must. Note that $[T, P_i]=0$, $[P_i, P_j]=0$ $(i, j=1, 2, 3)$.

The nonlinear wave equation (2) admits the same Lie-symmetry vector fields as equation (1).

The conservation laws can be derived by using the Cartan fundamental form. Again we use the jet bundle formalism.

Let us briefly describe the approach. Let M be an oriented manifold of dimension m, with local coordinates x_i and volume m-form Ω given in these coordinates by $\Omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$. Let N be an n-dimensional manifold with local coordinates u_j and let (E, π, M) be a fiber bundle with fiber N. The k jet bundle of local sections of (E, π, M) is denoted by $J^k(E)$.

We have $M = \mathscr{R}^4$, $N = \mathscr{R}$, and $(E, \pi, M) \equiv (M \times N, pr_1, M)$. Since $n = 1$, we put $u \equiv u_1$. Let (x_i, u) be a coordinate system on E and (x_i, u, u_i) the corresponding coordinates on $J^1(E)$. The Cartan fundamental form (a 4-form) defined on $J^1(E)$ is given by

$$\Theta = \left(L - \sum_{i=1}^{4} \frac{\partial L}{\partial u_i} u_i \right) \Omega + \sum_{i=1}^{4} \frac{\partial L}{\partial u_i} du \wedge \left(\frac{\partial}{\partial x_i} \rfloor \Omega \right) \tag{18}$$

where $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, $\rfloor$ denotes the contraction, and $L := J^1(E) \to \mathscr{R}$. In physics L is called the Lagrangian density.

We can introduce the quantity

$$H(u, p_i) = \frac{\partial L}{\partial u_4} u_4 - L \tag{19}$$

Table I. Commutation Relations

	R_{12}	R_{23}	R_{31}	L_1	L_2	L_3	S	I_1	I_2	I_3	I_4
T	0	0	0	P_1	P_2	P_3	T	$2L_1$	$2L_2$	$2L_3$	$2S$
P_1	P_2	0	$-P_3$	T	0	0	P_1	$2S$	$-2R_{21}$	$2R_{31}$	$2L_1$
P_2	$-P_1$	P_3	0	0	T	0	P_2	$2R_{12}$	$2S$	$-2R_{23}$	$2L_2$
P_3	0	$-P_2$	P_1	0	0	T	P_3	$-2R_{31}$	$2R_{12}$	$2S$	$2L_3$
R_{12}		$-R_{31}$	R_{23}	$-L_2$	L_1	0	0	$-I_2$	I_1	0	0
R_{23}			$-R_{12}$	0	$-L_3$	L_2	0	0	$-I_3$	I_2	0
R_{31}				L_3	0	$-L_1$	0	I_3	0	$-I_1$	0
L_1					R_{12}	$-R_{31}$	0	I_4	0	0	I_1
L_2						R_{23}	0	0	I_4	0	I_2
L_3							0	0	0	I_4	I_3
S								I_1	I_2	I_3	I_4
I_1									0	0	0
I_2										0	0
I_3											0

with the coordinates (x_i, u, p_i), where $p_i := \partial L/\partial u_i$. The quantity H is called the Hamiltonian density.

For the linear wave equation we have

$$L(u_i) = \tfrac{1}{2}(-u_1^2 - u_2^2 - u_3^2 + u_4^2) \tag{20}$$

and for the nonlinear wave equation

$$L'(u, u_i) = \tfrac{1}{2}(-u_1^2 - u_2^2 - u_3^2 + u_4^2) - \tfrac{1}{4}u^4 \tag{21}$$

The Hamiltonian density for the nonlinear wave equation (2) is given by

$$H(u, p_i) = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) + \tfrac{1}{4}u^4 \tag{22}$$

The Cartan fundamental form for the nonlinear wave equation (2) takes the form

$$\Theta = \left(L' - \sum_{i=1}^{4} \frac{\partial L'}{\partial u_i} u_i\right)\Omega + \sum_{i=1}^{4} (-1)^{i+1} \frac{\partial L'}{\partial u_i} du \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4 \tag{23}$$

The nonlinear wave equation (2) can be derived from the condition

$$s^*(Z \rfloor d\Theta) = 0 \tag{24}$$

where Z denotes the vertical vector field

$$Z = \tilde{Z}\frac{\partial}{\partial u} \tag{25}$$

and $s: M \to E$ is the section. We find

$$\begin{aligned} Z \rfloor d\Theta = \tilde{Z}(&u^3\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - du_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &+ dx_1 \wedge du_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge du_3 \wedge dx_4 \\ &- dx_1 \wedge dx_2 \wedge dx_3 \wedge du_4) \end{aligned} \tag{26}$$

Therefore equation (24) gives the nonlinear wave equation (2).

Since

$$L_K \Theta = d\xi \tag{27}$$

for the symmetry vector fields K given by equation (17), we obtain the conservation laws from $s^*(\xi - K \rfloor \Theta)$.

Finally we mention that Lie-Bäcklund vector fields cannot be found for the nonlinear wave equation (2) (Steeb, 1984). Moreover, the nonlinear wave equation (2) does not pass the Painlevé test [see Steeb and Euler (1988) for more details]. Owing to the symmetry vector field S, we know that equation (2) is scale invariant under $x_i \to \varepsilon^{-1}x_i$, $u \to \varepsilon u$. Therefore the Hamiltonian density scales like (Steeb and Louw, 1986)

$$H(\varepsilon u, \varepsilon^2 p_i) = \varepsilon^4 H(u, p_i) \tag{28}$$

so that $r=4$ and $r=-1$ are the so-called resonances. At the resonance $r=4$ the compatibility condition is not satisfied.

With the help of the Lie symmetry vector fields we can now also construct the similarity variables and the similarity ansatz. The similarity ansatz leads to ordinary differential eqquations. For example, the group-theoretic reduction of the nonlinear equation with the help of the space-time translation gives a nonlinear ordinary differential equation which can be solved with the help of elliptic functions. By making use of the rotation group, the group-theoretic reduction gives a nonlinear ordinary differential equation which cannot be solved exactly [see Steeb *et al.* (1985) for more details].

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